Numerical Solutions for the Composite Fractional Oscillation Equation by Using Fractional Difference Method

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Abstract

In this paper, Grünwald-Letnikov fractional derivatives, Riemann-Liouville fractional derivatives and Caputo fractional derivatives are introduced. The fractional difference method is applied for solving linear ordinary fractional differential equations of fractional order α. This method can be used for obtaining approximate solutions of fractional differential equations in different types. The composite fractional oscillation equation $(1 < \alpha \le 2)$ is solved by using this method.

Keywords: Grünwald-Letnikov, Riemann-Liouville, Caputo, Composite fractional oscillation, Fractional order

Introduction

We consider the numerical solution of linear fractional differential equation of the form

$$\frac{d^m u}{dt^m} - a\frac{d^\alpha u}{dt^\alpha} - bu = f(t),\ \ t > 0,\ \ m-1 < \alpha \le m, \tag{1}$$

subject to the initial conditions

$$u^{(j)}(0) = c_j, \qquad j = 0,1,2,\ldots,m-1, \tag{2}$$

where c_j, a and b are arbitrary constants and u(t) is assumed to be a causal function of time vanishing for $t < 0$. We refer to 6(1) as to the composite fractional oscillation equation in the cases $1 < \alpha \le 2$, $m = 2$.

Fractional Derivatives

Grünwald-Letnikov fractional derivatives

Let us consider a continuous real valued function $y = f(t)$ with step size h. The first order derivative of the function $f(t)$ is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0}\frac{f(t) - f(t-h)}{h}.$$

This equation can be used to form the second order derivative:

$$f''(t) = \frac{d^2 f}{dt^2} = \lim_{h\to 0}\frac{f(t) - 2f(t-h) + f(t-2h)}{h^2},$$

and the third order derivative:

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$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}.$$

The general form of an α^{th} derivative can be formed with induction:

$$f^{(\alpha)}(t) = \frac{d^\alpha f}{dt^\alpha} = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh), \tag{3}$$

where $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)L\ (\alpha - r + 1)}{r!}$ is the usual notation for the binomial coefficient.

Let us consider the following expression generalizing the functions:

$$f_h^{(\alpha)}(t) = \frac{1}{h^\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh), \tag{4}$$

where α is an arbitrary integer number and n is also integer.

For $\alpha \le n,$ we have

$$f^{(\alpha)}(t) = \frac{d^\alpha f}{dt^\alpha} = \lim_{h\to 0} f_h^{(\alpha)}(t),$$

because all the coefficients in the numerator after $\binom{\alpha}{\alpha}$ are equal to zero.

Let us consider the negative value of α and denote

$$\begin{bmatrix} \alpha \\ r \end{bmatrix} = \frac{\alpha(\alpha+1)L\ (\alpha + r - 1)}{r!}.$$

Then $\binom{-\alpha}{r} = \frac{-\alpha(-\alpha-1)L\ (-\alpha - r + 1)}{r!} = (-1)^r \begin{bmatrix} \alpha \\ r \end{bmatrix}$

Thus, (4) becomes

$$f_h^{(-\alpha)}(t) = \frac{1}{h^{-\alpha}} \sum_{r=0}^{n} \begin{bmatrix} \alpha \\ r \end{bmatrix} f(t - rh), \tag{5}$$

where α is positive integer number. If n is fixed, then $f_h^{(-\alpha)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. We have to suppose that $n \to \infty$ as $h \to 0$. We can take $h = \frac{t-a}{n}$, where a and t are terminals and a is a real constant.

Consider the limit value either finite or infinite of function $f_h^{(-\alpha)}(t)$ which will denote as

$${}_aD_t^{-\alpha}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-\alpha)}(t).$$

Thus $${}_aD_t^{-\alpha}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} \frac{1}{h^{-\alpha}}\sum_{r=0}^{n}\begin{bmatrix}\alpha\\ r\end{bmatrix} f(t-rh). \tag{6}$$

The derivative of an integer order n of the continuous function $f(t)$ is particular cases of the general expression

$${}_aD_t^{\alpha}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-\alpha}\sum_{r=0}^{n}(-1)^r\begin{pmatrix}\alpha\\ r\end{pmatrix} f(t-rh) \tag{7}$$

which represents the derivative of order m (m is a positive integer) if $\alpha = m$ and the m-fold integral if $\alpha = -m$. For negative value of α, we can form something equivalent to an integral:

$${}_aD_t^{-\alpha}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{\alpha}\sum_{r=0}^{n}\begin{bmatrix}\alpha\\ r\end{bmatrix} f(t-rh)$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau. \tag{8}$$

If the derivative $f'(t)$ is continuous in [a, t], then the integrating by parts, we can write (8) in the form

$${}_aD_t^{-\alpha}f(t) = \frac{f(a)(t-a)^{\alpha}}{G(\alpha+1)} + \frac{1}{G(\alpha+1)}\int_a^t (t-\tau)^{\alpha} f'(\tau)\,d\tau. \tag{9}$$

If the function $f(t)$ has $m+1$ continuous derivatives, then we have

$${}_aD_t^{-\alpha}f(t) = \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{\alpha+k}}{G(\alpha+k+1)} + \frac{1}{G(\alpha+m+1)}\int_a^t (t-\tau)^{\alpha+m} f^{(m+1)}(\tau)\,d\tau. \tag{10}$$

Let us consider $\alpha > 0$ in (7), we have

$${}_aD_t^{\alpha}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(\alpha)}(t) = \lim_{\substack{h\to 0\\ nh=t-a}} \frac{1}{h^{\alpha}}\sum_{r=0}^{n}(-1)^r\begin{pmatrix}\alpha\\ r\end{pmatrix} f(t-rh). \tag{11}$$

Using the binomial coefficient

$$\begin{pmatrix}\alpha\\ r\end{pmatrix} = \begin{pmatrix}\alpha-1\\ r\end{pmatrix} + \begin{pmatrix}\alpha-1\\ r-1\end{pmatrix}, \tag{12}$$

we can write

$$f_h^{(\alpha)}(t) = \frac{1}{h^{\alpha}}\sum_{r=0}^{n}(-1)^r\begin{pmatrix}\alpha-1\\ r\end{pmatrix} f(t-rh) + \frac{1}{h^{\alpha}}\sum_{r=1}^{n}(-1)^r\begin{pmatrix}\alpha-1\\ r-1\end{pmatrix} f(t-rh),$$

$$f_h^{(\alpha)}(t) = h^{-\alpha}\sum_{r=0}^{n}(-1)^r\binom{\alpha-1}{r}f(t-rh) + h^{-\alpha}\sum_{r=0}^{n-1}(-1)^{r+1}\binom{\alpha-1}{r}f(t-(r+1)h),$$

$$f_h^{(\alpha)}(t) = (-1)^n\binom{\alpha-1}{n}h^{-\alpha}f(a) + h^{-\alpha}\sum_{r=0}^{n-1}(-1)^r\binom{\alpha-1}{r}Vf(t-rh), \qquad (13)$$

where we denote $Vf(t-rh) = f(t-rh) - f(t-(r+1)h)$. The $Vf(t-rh)$ is a first-order backward difference of the function $f(\tau)$ at the point $\tau = t - rh$.

Applying the binomial coefficient repeatedly m times, (13) becomes

$$f_h^{(\alpha)}(t) = \sum_{k=0}^{m}(-1)^{n-k}\binom{\alpha-k-1}{n-k}h^{-\alpha}\Delta^k f(a+kh)$$

$$+ h^{-\alpha}\sum_{r=0}^{n-m-1}(-1)^r\binom{\alpha-m-1}{r}\Delta^{m+1}f(t-rh). \qquad (14)$$

We evaluate the limit of the k^{th} term in the first sum in (14):

$$\lim_{\substack{h\to 0\\ nh=t-a}}\sum_{k=0}^{m}(-1)^{n-k}\binom{\alpha-k-1}{n-k}h^{-\alpha}\Delta^k f(a+kh) = \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{G(-\alpha+k+1)}, \qquad (15)$$

and the limit of the second sum in (14):

$$\lim_{\substack{h\to 0\\ nh=t-a}} h^{-\alpha}\sum_{r=0}^{n-m-1}(-1)^r\binom{\alpha-m-1}{r}\Delta^{m+1}f(t-rh) = \frac{1}{\Gamma(-\alpha+m+1)}\int_a^t (t-\tau)^{m-\alpha}f^{(m+1)}(\tau)\,d\tau. \quad (16)$$

Using (15) and (16), we finally obtain

$${}_aD_t^{\alpha}f(t) = \lim_{\substack{h\to\infty\\ nh=t-a}} f_h^{(\alpha)}(t)$$

$$= \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+m+1)}\int_a^t (t-\tau)^{m-\alpha}f^{(m+1)}(\tau)\,d\tau. \qquad (17)$$

It is called Grünwald-Letnikov fractional derivative and has been obtained under the assumption that the derivatives $f^{(k)}(t),\ (k = 1, 2, 3, \ldots, m+1)$ are continuous in the closed interval [a,t] and m is an integer number satisfying the condition $m > \alpha - 1$. The smallest possible value for m is determined by the inequality $m < \alpha < m+1$.

For $1 < \alpha \le 2$, with the lower terminal $a = 0$ of the function $f(t)$, which is bounded at $t = 0$, (17) becomes the form:

$$ {}_0D_t^{\alpha} f(t) = \frac{f(0)t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \quad (18) $$

Let $\alpha = \frac{1}{2}$, $f(t) = t$. Then applying (18), we get

$$ {}_0D_t^{\frac{1}{2}} f(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-\tau)^{\frac{-1}{2}} d\tau = \frac{2\sqrt{t}}{\sqrt{\pi}}. $$

Riemann-Liouville fractional derivatives

Suppose that $\alpha > 0$, $t > a$, $\alpha, a, t \in R$. Then

$$ D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, & n-1 < \alpha < n, \text{for } n \in N \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \text{for } n \in N \end{cases} \quad (19) $$

is called the Riemann-Liouville fractional derivative or Riemann-Liouville fractional differential operator of order α.

Caputo's fractional derivatives

Suppose that $\alpha > 0$, $t > a$, $\alpha, a, t \in R$. Then

$$ D_*^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 < \alpha < n, \text{for } n \in N \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \text{for } n \in N \end{cases} \quad (20) $$

is called the Caputo fractional derivative (or) Caputo fractional differential operator of order α.

The power function

The Riemann-Liouville fractional derivative of the power function satisfies

$$ D^{\alpha} t^p = \frac{\Gamma(p+1}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \text{ where } n-1 < \alpha < n, \ p > -1, \ p \in R. $$

The Caputo fractional derivative of the power function satisfies

$$D_*^{\alpha} t^p = \begin{cases} \dfrac{\Gamma(p+1}{\Gamma(p-\alpha+1)} t^{p-\alpha} = D^{\alpha} t^p, \ n-1<\alpha<n, \ p>n-1, \ p\in R, \\ 0, \qquad\qquad\qquad\qquad n-1<\alpha<n, \ p\le n-1, \ p\in N. \end{cases}$$

Fractional Difference Method

We define the fractional derivative in the Grünwald-Letnikov sense as

$$D^{\alpha}u(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^j \binom{\alpha}{j} u(t-jh), \tag{21}$$

where [t] means the integer part of t and h is the step size.

The definition of operator in the Grüwald-Letnikov sense equation (21) is equivalent to the definition of operator in the Riemann-Liouville. Approximating the fractional derivative in (1) by (21), we obtain the following approximation for (1):

$$\sum_{j=0}^{m} C_j^m u_{m-j} - a \sum_{j=0}^{[t_m/h]} C_j^{\alpha} u_{m-j} - bu_m = f_m, \ (m=1,2,3,...), \tag{22}$$

where $t_m = mh$, $u_m = u(t_m)$ and C_j^{α} are Grüwald-Letnikov coefficients defined as

$$C_j^{\alpha} = h^{-\alpha}(-1)^j \binom{\alpha}{j}, \qquad (j=0,1,2,...). \tag{23}$$

We have

$$C_0^{\alpha} = h^{-\alpha} \quad \text{for } j = 0,$$

$$C_1^{\alpha} = -\alpha h^{-\alpha} = -\alpha C_0^{\alpha} = (1-\frac{1+\alpha}{1})C_0^{\alpha} \quad \text{for } j = 1,$$

$$C_2^{\alpha} = (-\alpha h^{-\alpha})(-\frac{(\alpha-1)}{2}) = (1-\frac{1+\alpha}{2})C_1^{\alpha} \quad \text{for } j = 2,$$

$$C_3^{\alpha} = (-1)\frac{(\alpha-2)}{3}(\frac{\alpha(\alpha-1)}{2\times 1}h^{-\alpha}) = (\frac{2-\alpha}{3})C_2^{\alpha} = (1-\frac{1+\alpha}{3})C_2^{\alpha} \quad \text{for } \quad j = 3.$$

By using the recurrence relationship

$$C_0^{\alpha} = h^{-\alpha}, \ C_j^{\alpha} = \left(1-\frac{1+\alpha}{j}\right)C_{j-1}^{\alpha}, \qquad (j=1,2,3,...). \tag{24}$$

For the case $1<\alpha\le 2$, m = 2, (22) becomes

$$\frac{u_m - 2u_{m-1} + u_{m-2}}{h^2} - ah^{-\alpha}u_m - a\sum_{j=1}^{m} C_j^{\alpha} u_{m-j} - bu_m = f_m,$$

$$u_m - 2u_{m-1} + u_{m-2} - ah^{-\alpha}u_m h^2 - ah^2 \sum_{j=1}^{m} C_j^{\alpha} u_{m-j} - bh^2 u_m = f_m h^2,$$

$$u_m - ah^{-\alpha}u_m h^2 - bh^2 u_m = f_m h^2 + 2u_{m-1} - u_{m-2} + ah^2 \sum_{j=1}^{m} C_j^{\alpha} u_{m-j},$$

$$[1-(b+ah^{-\alpha})h^2]u_m = 2u_{m-1} - u_{m-2} + ah^2 \sum_{j=1}^{m} C_j^{\alpha} u_{m-j} + f_m h^2.$$

Hence

$$u_0 = c_0, \quad \frac{u_1 - u_0}{h} = c_1,$$

$$u_m = \frac{2u_{m-1} - u_{m-2} + ah^2 \sum_{j=1}^{m} C_j^{\alpha} u_{m-j} + h^2 f_m}{1 - h^2(b + ah^{-\alpha})} \quad (m = 2,3,4,...). \tag{25}$$

These are the numerical solution algorithms for the composite fractional oscillation equation in the case $1 < \alpha \le 2$, $m = 2$.

Illustrations

Consider the following composite fractional oscillation equation

$$\frac{d^2u}{dt^2} - a\frac{d^{\alpha}u}{dt^{\alpha}} - bu = 8, \quad t > 0, 1 < \alpha \le 2, \tag{26}$$

subject to the initial conditions

$$u(0) = 0, \; u'(0) = 0. \tag{27}$$

By using the fractional difference method (25), we obtain the following solution:

$$u_m = \frac{2u_{m-1} - u_{m-2} - h^2 \sum_{j=1}^{m} C_j^{\alpha} u_{m-j} + 8h^2}{1 + h^2(1 + h^{-\alpha})}, \; (m = 2,3,4,... \;). \tag{28}$$

Giving $a = b = -1$ and $h = 0.01$, we get

$$u_m = \frac{2u_{m-1} - u_{m-2} - (0.01)^2 \sum_{j=1}^{m} C_j^{\alpha} u_{m-j} + 0.0008}{1 + (0.01)^2(1 + (0.01)^{-1.5})}, \; (m = 2,3,... \;). \tag{29}$$

The following table (Table 1) shows the approximate solution for the composite fractional oscillation equation (26) obtained the different values of fractional order α ($\alpha = 1.25$, $\alpha = 1.5$ and $\alpha = 1.75$), $a = b = -1$ and step size $h = 0.01$, by using the fractional difference method . Then we truncate the solution up to 100 terms.

Table 1

t	Fractional Difference Method		
	α = 1.25	α = 1.5	α = 1.75
0.0	0.0	0.0	0.0
0.1	0.0330848	0.0299729	0.0247897
0.2	0.1324928	0.1184009	0.0988386
0.3	0.2893951	0.2575560	0.2176142
0.4	0.4966003	0.4420972	0.3781759
0.5	0.7477783	0.6675843	0.5780095
0.6	1.0371718	0.9300431	0.8147500
0.7	1.3594531	1.2257758	1.0860717
0.8	1.7096454	1.5512656	1.3896378
0.9	2.0830754	1.9031262	1.7230762
1.0	2.4753462	2.2780731	2.0839694

Conclusion

It is found that the fractional difference method can be applied to get the approximate solutions of the composite fractional oscillation equations of fractional order $\alpha = 1.25$, $\alpha = 1.5$ and $\alpha = 1.75$ by using the MATLAB programming.

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